Quantum phase transitions of fractons
(a fractal scaling theory for the FQHE)

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Abstract

We consider the quantum phase transitions of fractons in correspondence with the quantum phase transitions of the fractional quantum Hall effect-FQHE. We have that the Hall states can be modelled by fractons, known as charge-flux systems which satisfy a fractal distribution function associated with a fractal von Neumann entropy. In our formulation, the universality classes of the fractional quantum Hall transitions, are considered as fractal sets of dual topological quantum numbers filling factors labelled by the Hausdorff dimension h ($1 < h < 2$) of the quantum paths of fractons. In this way we have defined, associated to these universality classes, a scaling exponent as $\kappa = 1/h$, such that when h runs into its interval of definition, we obtain $1 \gtrsim \kappa \gtrsim 0.5$. The behavior of this scaling exponent, topological in character, is in agreement with some experimental values claimed in the literature. Thus, according to our approach we have a fractal scaling theory for the FQHE which distinguishes diverse universality classes for the fractional quantum Hall transitions.

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I. INTRODUCTION

Quantum phase transitions in complex systems [1], like fractional quantum Hall effect-FQHE, have been object of great attention in the last fews years focussed on a better understanding of these two-dimensional strongly correlated electron systems. This type of transition occurs in a quantum critical point at $T = 0$ when some parameter of the Hamiltonian is varied. The transition is characterized by an order parameter which is zero in the disordered phase, and non-zero and non-unique in the ordered phase. In that critical point divergences of the correlation lenght and the correlation time occur and the observables present power laws on the external parameters. In this way some exponents characterize the critical behavior of a particular phase transition. Now, in the literature, theoretical and experimental analysis of the FQHE have considered the Hall transitions in the same universality class with scaling exponent $\zeta = 0.45 \pm 0.05$ and a universal critical exponent $\gamma = 2.3 \pm 0.1$ [2,3]. On the other hand, there is a suggestion that microscopic details can imply in different universality classes for the FQHE [3]. In contrast, taking into account only global properties (as the modular group and the fractal dimension of quantum paths), our fractal scaling theory for the FQHE shows us subtle differences between fractal sets of filling factors. This theory signalizes distinct universality classes for the fractional quantum Hall phase transitions which in some sense is in agreement with other experimental results. As we can verify [1], quantum phase transitions remain with too much open questions to be investigated. Thus, we propose to discuss, according our view, a controversial point on the universality class of the FQHE.

The paper is organized as follows, in the second section we review our fractal approach to the fractional spin particles and we make a connection with the FQHE; in the third section we discussed the entanglement content of the Hall states in terms of an entanglement measure for the universal classes of fractons and we extract the result that Hall states of a given universality class have the same amount of entanglement; in the fourth section we introduced our fractal scaling theory for the FQHE which gives us a new perspective for that phenomenon; and in the last section we finish with some remarks.

II. UNIVERSAL CLASSES OF FRACTONS

The mathematical ideas about fractal geometry [4] have been applied in various contexts of physics, as in nonlinear and nonequilibrium phenomena [5]. Here, in this Letter, we show that the FQHE [6] has a fractal-like structure such that the universality classes of the quantum Hall transitions constitute fractal sets labelled by the Hausdorff dimension defined within the interval $1 < h < 2$ and associated with fractal curves (continuous functions but nowhere differentiable) of objects called fractons. These ones are charge-flux systems defined in two-dimensional multiply connected space and they carry rational or irrational values of spin. The topological meaning of the dual filling factors which characterizes the FQHE comes from the connection between the fractal dimension of the quantum paths of fractons and these quantum numbers. Thus, we have introduced *a new geometric insight*

for understanding that phenomenon [1].

Topological quantum numbers are insensitive to the imperfections of the systems and so the *fractal dimension* as a topological invariant makes robust the FQHE properties. By contrast, in the literature, some authors have considered the concepts of Chern numbers and Fredholm indices for the integer quantum Hall effect, but for the FQHE this research line is an open challenge. The Fredhom index is related to the creation and annihilation operators of many-body quantum mechanics and in some cases coincides with the Chern number [7]. The mathematical mechanism responsible for the FQHE has been also considered using some techniques of noncommutative geometry in connection with twisted higher index theory of elliptic operators on orbifold (its definition generalizes the idea of a manifold) covering spaces of compact good orbifolds, where the topological nature of the Hall condutance is stable under small deformations of a Hamiltonian, with the interaction simulates by the negative curvature of the hyperbolic structure of the model. The twisted higher index can be a fraction when the orbifold is not smooth and the topological invariant is namely the orbifold Euler characteristic [8]. The connection between the FQHE and topological Chern-Simons field theories is another route of investigation [9]. Therefore, our fractal approach offers *a possible alternative* for understanding some deeper mathematical and physical features underlying the FQHE [10–18].

Fractons satisfy a fractal distribution function associated with a fractal von Neumann entropy and they are classified in universal classes of particles or quasiparticles [2]. Our formulation was introduced in [10] and in particular, we have found an expression which relates the fractal dimension h and the spin s of the particles, $h = 2 - 2s$, $0 < s < \frac{1}{2}$. This result is analogous to the fractal dimension formula of the graph of the functions, in the context of the fractal geometry, and given by: $\Delta(\Gamma) = 2 - H$, where H is known as Hölder exponent, with $0 < H < 1$ [4]. The bounds of the fractal dimension $1 < \Delta(\Gamma) < 2$ needs to be obeyed in order for a function to be a fractal, so the bounds of our parameter h are defined such that, for $h = 1$ we have fermions, for $h = 2$ we have bosons and for $1 < h < 2$ we have fractons. The Hölder exponent characterizes irregular functions which appears in diverse physical systems. For instance, in the Feynman path integral approach of the quantum mechanics, the fractal character of the quantum paths was just observed [19].

The fractal properties of the quantum paths can be extracted from the propagators of the particles in the momentum space [10,20] and so our expression relating h and s can once more be justified. On the other hand, the physical formula introduced by us, when we consider the spin-statistics relation $\nu = 2s$, is written as $h = 2 - \nu$, $0 < \nu < 1$. This way, a fractal spectrum was defined taking into account a mirror symmetry:

$$h - 1 = 1 - \nu, \quad 0 < \nu < 1; \qquad h - 1 = \nu - 1, \qquad 1 < \nu < 2;$$

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[1]Another fractal formulation in connection with the FQHE was discussed in [35,36].

[2]The universal classes of fractons are defined in the same way that fermions and bosons constitute universal classes of particles with semi-integer and integer values of spin, satisfying the fermionic and bosonic distribution functions, respectively. Thus, we have universal classes of particles with rational or irrational values of spin satisfying a specific fractal distribution function.

$$h - 1 = 3 - \nu, \quad 2 < \nu < 3; \qquad h - 1 = \nu - 3, \qquad 3 < \nu < 4; \; etc. \tag{1}$$

The statistical weight for these classes of fractons is given by [10]

$$\mathcal{W}[h, n] = \frac{[G + (nG - 1)(h - 1)]!}{[nG]! \, [G + (nG - 1)(h - 1) - nG]!} \tag{2}$$

and from the condition of the entropy be a maximum, we obtain the fractal distribution function

$$n[h] = \frac{1}{\mathcal{Y}[\xi] - h}. \tag{3}$$

The function $\mathcal{Y}[\xi]$ satisfies the equation

$$\xi = \left\{ \mathcal{Y}[\xi] - 1 \right\}^{h-1} \left\{ \mathcal{Y}[\xi] - 2 \right\}^{2-h}, \tag{4}$$

with $\xi = \exp\left\{ (\epsilon - \mu)/KT \right\}$.

We understand the fractal distribution function as a quantum-geometrical description of the statistical laws of nature, since the quantum path is a fractal curve (this property was noted by Feynman) and this reflects the Heisenberg uncertainty principle. The Eq.(3) embodies nicely this subtle information about the quantum paths associated with the particles.

We can obtain for any class its distribution function considering the Eqs.(3,4). For example, the universal class $h = \frac{3}{2}$ with distinct values of spin $\left\{ \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots \right\}_{h=\frac{3}{2}}$, has a specific fractal distribution

$$n\left[\frac{3}{2} \right] = \frac{1}{\sqrt{\frac{1}{4} + \xi^2}}. \tag{5}$$

This result coincides with another one of the literature of fractional spin particles for the statistical parameter $\nu = \frac{1}{2}$ [21], however our interpretation is completely distinct. This particular example, shows us that the fractal distribution is the same for all the particles into the universal class labelled by h and with different values of spin. Thus, we emphasize that in our formulation the spin-statistics connection is valid for such fractons. The authors in [21] never make mention to this possibility. Therefore, our results give another perspective for the fractional spin particles or anyons [9]. On the other hand, we can obtain straightforward the Hausdorff dimension associated to the quantum paths of the particles with any value of spin.

We also have

$$\xi^{-1} = \left\{ \Theta[\mathcal{Y}] \right\}^{h-2} - \left\{ \Theta[\mathcal{Y}] \right\}^{h-1}, \tag{6}$$

where

$$\Theta[\mathcal{Y}] = \frac{\mathcal{Y}[\xi] - 2}{\mathcal{Y}[\xi] - 1} \tag{7}$$

is the single-particle partition function. We verify that the classes h satisfy a duality symmetry defined by $\tilde{h} = 3 - h$. So, fermions and bosons come as dual particles. As a consequence, we extract a fractal supersymmetry which defines pairs of particles $\left(s, s + \frac{1}{2}\right)$. This way, the fractal distribution function appears as a natural generalization of the fermionic and bosonic distributions for particles with braiding properties. Therefore, our approach is a unified formulation in terms of the statistics which each universal class of particles satisfies, from a unique expression we can take out any distribution function. In some sense , we can say that fermions are fractons of the class $h = 1$ and bosons are fractons of the class $h = 2$.

The free energy for particles in a given quantum state is expressed as

$$\mathcal{F}[h] = KT \ln \Theta[\mathcal{Y}]. \tag{8}$$

The fractal von Neumann entropy per state in terms of the average occupation number is given as [10,11]

$$\mathcal{S}_G[h, n] = K \left[[1 + (h - 1)n] \ln \left\{ \frac{1 + (h - 1)n}{n} \right\} - [1 + (h - 2)n] \ln \left\{ \frac{1 + (h - 2)n}{n} \right\} \right] \tag{9}$$

and it is associated with the fractal distribution function Eq.(3).

Now, as we can check, each universal class h of particles, within the interval of definition has its entropy defined by the Eq.(9). Thus, for fractons of the self-dual class $\left\{ \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots \right\}_{h = \frac{3}{2}}$, we obtain

$$\mathcal{S}_G \left[\frac{3}{2} \right] = K \left\{ (2 + n) \ln \sqrt{\frac{2 + n}{2n}} - (2 - n) \ln \sqrt{\frac{2 - n}{2n}} \right\}. \tag{10}$$

The Fermi and Bose statistics (and the respective entropies, free energies) associated to the universal classes of the fermions and bosons are recuperated promptly.

We have also introduced the topological concept of fractal index, which is associated with each class. As we saw, h is a geometrical parameter related to the quantum paths of the particles and so, we define [12]

$$i_f[h] = \frac{6}{\pi^2} \int_{\infty(T=0)}^{1(T=\infty)} \frac{d\xi}{\xi} \ln \left\{ \Theta[\mathcal{Y}(\xi)] \right\}. \tag{11}$$

For the interval of the definition $1 \leq h \leq 2$, there exists the correspondence $0.5 \leq i_f[h] \leq 1$, which signalizes a connection between fractons and quasiparticles of the conformal field theories, in accordance with the unitary $c < 1$ representations of the central charge. Thus, we have established a connection between fractal geometry and number theory, given that the dilogarithm function appears in this context, besides another branches of mathematics [22].

Such ideas can be applied in the context of the FQHE. This phenomenon is characterized by the filling factor parameter f, and for each value of f we have the quantization of Hall resistance and a superconducting state along the longitudinal direction of a planar system of electrons, which are manifested by semiconductor doped materials, i.e., heterojunctions under intense perpendicular magnetic fields and lower temperatures [6].

The parameter f is defined by $f = N\frac{\phi_0}{\phi}$, where N is the electron number, ϕ_0 is the quantum unit of flux and ϕ is the flux of the external magnetic field throughout the sample. The spin-statistics relation is given by $\nu = 2s = 2\frac{\phi\prime}{\phi_0}$, where $\phi\prime$ is the flux associated with the charge-flux system which defines the fracton (h, ν). According to our approach there is a correspondence between f and ν, numerically $f = \nu$. This way, we verify that the filling factors experimentally observed appear into the classes h and from the definition of duality between the equivalence classes, we note that the FQHE occurs in pairs of these dual topological quantum numbers.

III. ENTANGLEMENT FRACTAL VON NEUMANN ENTROPY

In [23] we have defined an entanglement measure for the universal classes of fractons in terms of the probability distribution p as:

$$\mathcal{E}[h, p] = \frac{1}{1 - (h - 1)p} \left\{ -p \log_2 p - (1 - p) \log_2(1 - p) \right\}, \tag{12}$$

where $0 \leq p \leq 1$, is the probability of the system to be in a microstate with entanglement between ocuppation-numbers of the modes considered empty, partially or completely filled.

The entanglement properties of fractons (Hall states) can be analyzed considering the Eq.(12) and we verify that the entanglement increases when we run in the interval $1 < h < 2$, for instance, $\mathcal{E}[h = 4/3] < \mathcal{E}[h = 3/2] < \mathcal{E}[h = 5/3]$. Given the sequence,

$$\cdots \rightarrow \left\{ \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \cdots \right\}_{h=\frac{4}{3}} \rightarrow \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \right\}_{h=\frac{3}{2}} \rightarrow \left\{ \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \cdots \right\}_{h=\frac{5}{3}} \rightarrow \cdots,$$

and the entanglement measure in terms of the filling factors

$$\mathcal{E}[2 - \nu, p] = \frac{1}{1 - (1 - \nu)p} \left\{ -p \log_2 p - (1 - p) \log_2(1 - p) \right\}, \tag{13}$$

with $0 < \nu < 1$, we obtain $\mathcal{E}[\nu = 2/3] < \mathcal{E}[\nu = 1/2] < \mathcal{E}[\nu = 1/3]$. For the other members of the classes we need to consider the fractal spectrum Eq.(1). This way, we verify that the Eq.(12) for the class h, is the same for all the members of the class and so, in terms of their entanglement content, different Hall states are equivalent. The understanding that something in this sense can be provided by a quantitative theory of entanglement for complex quantum systems was envisaged by Osborne-Nielsen in [24]. Therefore, we have obtained a result, in the context of the FQHE, which just realizes this perception. Observe that our approach gives information about the entanglement for any possible wave function associated with a specific value of the filling factor. In another route we can consider the LLL for fractons, i.e. if the temperature is sufficiently low and $\epsilon < \mu$, we can check that the mean ocuppation number Eq.(3) is given by $n = \frac{1}{2-h}$, and so the fractal parameter h regulates the number of particles in each quantum state. At $T = 0$ and $\epsilon > \mu$, $n = 0$ if $\epsilon > \epsilon_F$ and $n = \frac{1}{2-h}$ if $\epsilon < \epsilon_F$, hence we get a step distribution, taking into account the Fermi energy ϵ_F and $h \neq 2$. We can check that for $h = \frac{4}{3}, \frac{3}{2}, \frac{5}{3}$ we obtain $n = \frac{3}{2}, \frac{2}{1}, \frac{3}{1}$, respectively. In the first case we have three particles for two states, in the second case two particles for one state and in the last case three particles for one state. So when we run in the interval

$1 < h < 2$ we gain more particles for each possible state. In some sense fractons can be understood as *quasifermions* when near the universal class $h = 1$ and as *quasibosons* when near the universal class $h = 2$. The entanglement of the FQHE increases because we have more particles (fractons) and less states. On the other hand, in terms of the filling factors, the average ocuppation number can be written as $n = \frac{1}{\nu}$, $0 < \nu < 1$; $n = \frac{1}{2-\nu}$, $1 < \nu < 2$; $n = \frac{1}{\nu-2}$, $2 < \nu < 3$; etc. The behavior of the step distribution confirms our former analysis: the ground state of the FQHE is a stronger entangled state and the entanglement between ocuppation-numbers of fractons in the LLL and the rest of the system shows us quantum correlations which can be quantified.

All these results agree with the entanglement properties of the Laughlin wave functions and those generated by the K-matrix [25]. On the other hand, the FQHE understood in terms of the composite fermions or composite bosons are non-entangled as observed in [26], so in contrast, fractons appear as a suitable system for study the quantum correlations of the FQHE. Thus the suggestion that ideas of the quantum information science can give insights for understanding some complex quantum systems [24] is manifested in our definition of entanglement measure for the universal classes of fractons. As we saw, the entanglement increases in the interval $1 < h < 2$ and this suggests fracton qubits as a physical resource for quantum computing. In the literature FQHE qubits associated with the geometrical characteristic of the fractional spin particles have been exploited [27]. The quantum Hall phase transitions discussed by us were obtained considering global properties as the modular symmetry and the Hausdorff dimension associated to the quantum paths of the particles, so some peculiarities of the FQHE, in particular, do not depend on the microscopic details of this strongly interacting system [10].

Here, we observe that our approach, in terms of fractal sets of dual topological filling factors, embodies the structure of the modular group as discussed in the literature [18,28] and the quantum Hall transitions satisfy some properties related with the Farey sequences of rational numbers. The transitions allowed are those generated by the condition $\mid p_2q_1 - p_1q_2 \mid = 1$, with $\nu_1 = \frac{p_1}{q_1}$ and $\nu_2 = \frac{p_2}{q_2}$. In [28] the properties of integer and fractional quantum Hall effect were considered in terms of a subgroup of the modular group $SL(2, \mathbf{Z})$, such that the group acts on the upper-half complex plane parameterised by a complex conductivity and so generates the phase diagram of the quantum Hall effect. The rules obtained there are the same that our formulation [10], but we have defined universality classes of the quantum Hall transitions in terms of fractal sets labelled by a fractal parameter. On the other one, our theoretical description consider the Hall states modelled by systems of fractons with a specific value of spin.

IV. FRACTAL SCALING THEORY FOR THE FQHE TRANSITIONS

As we discussed earlier, the universality classes of the fractional quantum Hall transitions have been considered as fractal sets of dual topological quantum numbers filling factors labelled by the Hausdorff dimension. Such approach has the fractal parameter h as a critical exponent for the correlation length associated with a scaling exponent defined as $\kappa = 1/h$. So, the members (possible Hall states characterized by the values of the filling factor) of a given class, for example, $\left\{ \frac{1}{3}, \frac{5}{3}, \cdots \right\}_{h=\frac{5}{3}}$ or $\left\{ \frac{2}{3}, \frac{4}{3}, \cdots \right\}_{h=\frac{4}{3}}$, share the same scaling exponent

$\kappa = 0.6$ and $\kappa = 0.75$, respectively. We can establish families of quantum Hall phase transitions as

$$\left\{\frac{1}{5}, \frac{9}{5}, \frac{11}{5}, \frac{19}{5}, \cdots\right\}_{h=\frac{9}{5}}^{\kappa=0.55} \rightarrow \left\{\frac{2}{9}, \frac{16}{9}, \frac{20}{9}, \frac{34}{9}, \cdots\right\}_{h=\frac{16}{9}}^{\kappa=0.56} \rightarrow \left\{\frac{1}{4}, \frac{7}{4}, \frac{9}{4}, \frac{15}{4}, \cdots\right\}_{h=\frac{7}{4}}^{\kappa=0.57} \rightarrow$$

$$\left\{\frac{2}{7}, \frac{12}{7}, \frac{16}{7}, \frac{26}{7}, \cdots\right\}_{h=\frac{12}{7}}^{\kappa=0.58} \rightarrow \left\{\frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3}, \cdots\right\}_{h=\frac{5}{3}}^{\kappa=0.60} \rightarrow \left\{\frac{2}{5}, \frac{8}{5}, \frac{12}{5}, \frac{18}{5}, \cdots\right\}_{h=\frac{8}{5}}^{\kappa=0.625} \rightarrow$$

$$\left\{\frac{3}{7}, \frac{11}{7}, \frac{17}{7}, \frac{25}{7}, \cdots\right\}_{h=\frac{11}{7}}^{\kappa=0.63} \rightarrow \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots\right\}_{h=\frac{3}{2}}^{\kappa=0.66} \rightarrow \left\{\frac{4}{7}, \frac{10}{7}, \frac{18}{7}, \frac{24}{7}, \cdots\right\}_{h=\frac{10}{7}}^{\kappa=0.70} \rightarrow$$

$$\left\{\frac{3}{5}, \frac{7}{5}, \frac{13}{5}, \frac{17}{5}, \cdots\right\}_{h=\frac{7}{5}}^{\kappa=0.71} \rightarrow \left\{\frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{10}{3}, \cdots\right\}_{h=\frac{4}{3}}^{\kappa=0.75} \rightarrow \left\{\frac{5}{7}, \frac{9}{7}, \frac{21}{7}, \frac{23}{7}, \cdots\right\}_{h=\frac{9}{7}}^{\kappa=0.77} \rightarrow$$

$$\left\{\frac{3}{4}, \frac{5}{4}, \frac{11}{4}, \frac{13}{4}, \cdots\right\}_{h=\frac{5}{4}}^{\kappa=0.80} \rightarrow \left\{\frac{7}{9}, \frac{11}{9}, \frac{25}{9}, \frac{29}{9}, \cdots\right\}_{h=\frac{11}{9}}^{\kappa=0.81} \rightarrow \left\{\frac{4}{5}, \frac{6}{5}, \frac{14}{5}, \frac{16}{5}, \cdots\right\}_{h=\frac{6}{5}}^{\kappa=0.82}.$$

We can see, in this sequence, that we have sets with values of scaling exponent $\kappa \sim 0.5, 0.6, 0.7, 0.8$. So, up to now, the experiments did not detect these subtle differences between distinct universality classes for the FQHE. When h runs into the interval of definition, the scaling exponent stays into $1 \gtrsim \kappa \gtrsim 0.5$. On the other hand, in the literature, values for the scaling exponent $\zeta \sim 0.45, 0.6, 0.77$ have been reported, with $\zeta = 1/z\gamma$, where z is a dynamical critical exponent and γ is a universal critical exponent. We observe that for adjacent quantum Hall phase transitions, the scaling exponents do not present appreciable difference between them. So, in the experiments, this is not detected and the scaling exponent appears with the same value for all the filling factors. *We believe that this is not the case.* We have here fine differences between distinct classes of Hall states. In [29] was verified a transition from the integer filling factor $\nu = 1$ to the fractional filling factor $\nu = 2/3$ with scaling exponent $\zeta = 0.77 \pm 0.02$ which coincides with our value of $\kappa = 0.75$ just for the class $h = 4/3$ where the filling factor considered appears. We enphasize that all the members of this class have the same scaling exponent. In this sense, these Hall states are in the same universality class. However, the transitions $3/5 - 2/3$, $1 - 4/3$, $5/3 - 2$ have been reported with scaling behavior $\zeta = 0.45 \pm 0.05$ and for the transition $2/3 - 1$, $\zeta = 0.77$ [30]. Also, for the transition $1/3$ to $2/5$, was found [31] the value of $\zeta = 0.43 \pm 0.02$ while via our approach we have that for the filling factor $1/3$ we obtain $\kappa = 0.6$ and for $\nu = 2/5$, $\kappa = 0.625$. In another experiment, a scaling exponent $\zeta = 0.5 \pm 0.1$ was found which is independent of material, density, mobility, experimental technique, temperature and the filling factor [32]. As we can see, our understanding of the quantum phase transitions, in this quantum complex system-FQHE, until now is not complete and controversial [33]. Thus, our fractal scaling theory signalizes some correspondence between the scaling exponent, κ, for quantum phase transitions of fractons and the scaling exponent, ζ, for fractional quantum Hall transitions. This gives us the opportunity of to distinguish universality classes for the FQHE and so we suggest to the experimentalists looking for this fractal scaling behavior in these two-dimensional strongly correlated electron systems.

Here we emphasize the topological character of the scaling exponent κ, because it is derived of a geometrical parameter associated to the fractal curves of the quantum particles.

As this reflects the Heisenberg uncertainty principle [3], quantum fluctuations are encoded in this concept of Hausdorff dimension associated to the quantum paths of fractons. Also, as the filling factors are topological quantum numbers in principle and this characteristic in our theoretical formulation is extracted from the connection between h and ν, via the fractal spectrum Eq.(1), we can think κ as a signature of some kind of topological phase transition between fractal sets of filling factors and so they are in correspondence with the quantum Hall phase transitions.

V. CONCLUSIONS

We have suggested in this article that the fractional quantum Hall phase transitions obey a fractal scaling theory which implies distinct universality classes for that phenomenon. According to our approach these transitions are characterized by a scaling exponent, κ, which is related with the fractal dimension, h, of the quantum paths of fractons. In this way the topological character of the scaling exponent signalizes transitions between fractal sets of filling factors which do not depend on the microscopic details of the physical system. These topological quantum phase transitions show, for values of the scaling exponent into the interval $1 \gtrsim \kappa \gtrsim 0.5$, correspondence with some experimental values. Besides these robust aspects, our approach via the concept of the entanglement measure for the universal classes of fractons has extracted the result that the Hall states within of the same universality class have equivalent entanglement. Also along of our discussion is implicit that our approach to the FQHE reproduces all experimental data and can predicting the occurrence of this phenomenon for other filling factors. These topological quantum numbers are obtained from first-principles and the quantum Hall states are modelled by fractons which carry rational or irrational values of spin and satisfy a fractal distribution function associated with a fractal von Neumann entropy. Therefore, the physical scenario takes place and the thermodynamical properties of such systems can be investigated.

We emphasize that our approach is supported by symmetry principles such as mirror symmetry behind the fractal spectrum, duality symmetry between the universal classes of particles, fractal supersymmetry, modular group behind the universality classes of the quantum Hall transitions. Besides these stronger arguments given that the symmetry groups have great importance for understanding the physical theories even before any dynamics, we have established a connection between physics, fractal geometry and number theory. Along this discussion we have obtained fractal sets of dual topological quantum numbers filling factors associated with the FQHE. The universality classes of the quantum Hall transitions were established and the fractal geometry of nature in this context is manifest in a simple and intuitive way.

Here, we observe a possible connection between ours results and a discussion in [34], which consider if the universality classes of a phase transition depends on the fractional statistics. For that, we have an affirmative answer according to our fractal approach to the

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[3]Quantum fluctuations are driven by the Heisenberg uncertainty principle and this one expresses the fractal character of the quantum paths.

fractional spin particles. Also, as the fractons obey the spin-statistics connection and we have verified a fractal supersymmetry between the universal classes of these objects, we stay with the intuition that we can advance a hypothesis on the analyticity of a given amplitude introduced in the context of some models studied in [34].

Finally, we believe that our fractal scaling theory contains some ideas which can be considered within the structure of some other theory ahead, such that be possible a more deeper understanding of the FQHE.

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